AXIALLY SYMMETRIC PROBLEM OF THE NONLINEAR THEORY OF ELASTICITY FOR AN INCOMPRESSIBLE MEDIUM

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1. We consider the axially symmetric strains in a body of rotation. We denote the cylindrical coordinates of points in the body in the initial state by X^A $(A = 1, 2, 3)$, where X^1 is taken along the axis of rotation, X^2 is the distance of points from the axis, and X^3 is a polar angle.

The position of points in the deformed body will be determined by the expanded coordinates x^i $(i = 1, 2, 3)$ in the same cylindrical coordinate system.

In the notation for tensor components, capital Latin letters will correspond to the original coordinates and small letters to the expanded coordinates. A covariant original derivative of the tensor $T::$ is denoted by coordinate indices after the comma $T::,_A$; for an expanded derivative the same indices used but with small letters, as $T::,_i$. Displacements of the body points during deformation are denoted by U^1, U^2 in the original coordinates, corresponding to X^1 and X^2 ; displacements U^3 perpendicular to the plane of the meridian are taken equal to zero. Thus, for the problem considered, the contravariant components of the vector of the medium displacement are represented by functions of the two parameters

$$U^1 = U^1 (X^1, X^2), \qquad U^2 = U^2 (X^1, X^2) \tag{1.1}$$

It is shown here that the incompressibility condition for the medium gives the possibility of expressing the unknown functions (1.1) by a certain function of two parameters (called here the displacement function), and of reducing the solution of the axially symmetric problem of the nonlinear theory of elasticity to the problem of finding a single function satisfying the equations of equilibrium and the boundary conditions.

2. Let α, β be the curvilinear coordinates in the plane X^1, X^2 . Let $\omega(\alpha,\beta)$ be a function having continuous derivatives up to the fourth order incl., and which satisfies condition

$$\Delta = 1 - \omega_{\alpha\beta}^2 + \omega_{\alpha\alpha}\omega_{\beta\beta} \neq 0, \qquad -\beta < \omega_\alpha < \beta \tag{2.1}$$

The Greek indices denote partial derivatives of ω

The incompressibility condition for the medium [1]

$$I = |\delta^A{}_B + U^A{}_{,B}| = 1 \tag{2.2}$$

is presented in the form

$$\left[\left(1+\frac{\partial U^1}{\partial X^1}\right)\left(1+\frac{\partial U^2}{\partial X^2}\right)-\frac{\partial U^1}{\partial X^2}\frac{\partial U^2}{\partial X^1}\right]\left(1+\frac{U^2}{X^2}\right)=1 \qquad (2.3)$$

We shall seek a solution of the incompletely defined differential equation (2.3) in the form of a family of functions of the two parameters. We proceed by a change of variables

$$z = U^1 + X^1, \qquad w = \frac{(U^2 + X^2)\,(U^2 + X^2)}{2} \qquad (2.4)$$

which gives Equation (2.3) in the form

$$\left(\frac{\partial z}{\partial X^1}\frac{\partial w}{\partial X^2}-\frac{\partial z}{\partial X^2}\frac{\partial w}{\partial X^1}\right)\frac{1}{X^2}=1 \qquad (2.5)$$

and we present [2] the solution of (2.5) with the aid of two functions $P(\alpha,\beta)$, $Q(\alpha,\beta)$ in the form

$$X_1 = \alpha + P, \qquad X^2 = \sqrt{2(\beta + Q)}, \qquad z = \alpha - P, \qquad w = \beta - Q \qquad (2.6)$$

By substitution of (2.6) into (2.5) we obtain Equation

$$P_\alpha + Q_\beta = 0 \qquad (2.7)$$

From this we have

$$P = \omega_\beta(\alpha, \beta), \qquad Q = -\omega_\beta(\alpha, \beta) \qquad (2.8)$$

Here $\omega(\alpha\ \beta)$ is an arbitrary function having continuous derivatives of the first and second order.

The assumption is made that

$$\Delta = (1 + P_\alpha)(1 + Q_\beta) - P_\beta Q_\alpha \neq 0 \qquad (2.9)$$

Consequently, the solution of differential equation (2.3) may be given in the form of Formulas

$$U^1 = -2\omega_\beta, \qquad U^2 = \sqrt{2(\beta + \omega_\alpha)} - \sqrt{2(\beta - \omega_\alpha)} \qquad (2.10)$$
$$X^1 = \alpha + \omega_\beta, \qquad X^2 = \sqrt{2(\beta - \omega_\alpha)}$$

Thus, the unknown displacements U^1 and U^2 are expressed as derivatives of a single function $\omega(\alpha\ \beta)$ of the two parameters.

We note that the representation (2.10) of the medium remains spatially Euclidean.

3. The compatibility conditions for strain in the axisymmetric problem may be obtained in a simpler form without having recourse to the curvature tensor of the medium.

If the displacements in the medium are continuous, then

$$\frac{\partial}{\partial X^1}\frac{\partial U^A}{\partial X^2} = \frac{\partial}{\partial X^2}\frac{\partial U^A}{\partial X^1} \qquad (A = 1, 2) \qquad (3.1)$$

and for compatibility of strain it is sufficient to require that the angle of rotation of any fiber in the meridional plane also be continuous. We obtain the continuity condition for the angles of rotation of the fibers.

As is well known [3], the components α_B in the expansion

$$a_{A,B} = \alpha_B a_A^{\circ} + \beta_B a_A \qquad (3.2)$$

of the covariant derivative of the vector field a on the surface determine the transverse vector of the field by vectors of the given field and by an additional field.

Besides, the transverse vectors of two fields differ by the gradient of the angle between the vectors of these fields [3], i.e.

$$\gamma_B = \alpha_B + \frac{\partial\varphi}{\partial X^B} \tag{3.3}$$

Here γ_B and α_B are the covariant coordinates of the transverse vectors of the fields; φ is the angle between the vectors of the two fields.

By using this property and by choice of the second covariant derivative of the angle φ, we obtain the continuity condition for rotation in the form

$$\gamma_{[B,A]} - \alpha_{[B,A]} = 0 \tag{3.4}$$

if γ_B and α_B are transverse vectors of certain material fibers, lying in the meridional plane before and after deformation, respectively. The choice of the tensor is shown by square brackets

$$\alpha_{[A,B]} = {}^1/_2\,[\alpha_{A,B} - \alpha_{B,A}]$$

Thus, for compatibility of axisymmetric strain, it is necessary and sufficient that (3.1) and (3.4) be satisfied identically.

4. For a given function $\omega(\alpha,\beta)$ each system of displacements corresponds to a certain stress field if only the stress – strain relations are known.

But such a stress field will not, generally speaking, satisfy the equilibrium conditions for an element of the medium, nor the boundary conditions.

The equilibrium conditions for an element of the medium are isolated from the geometrically assumed, statically possible strain systems given by $\omega(\alpha,\beta)$, which determine the stress – strain relation expressing the mechanical properties of the medium.

Those equations isolated from the system of geometrically possible strain systems, satisfying the equilibrium equations of the medium with a certain potential strain energy density Σ, we call the equations of compatibility of stress and strain. The equations of compatibility for isotropic materials will be obtained here. It should be noted that analogous equations of compatibility may also be obtained for other types of incompressible materials.

The equations of equilibrium in an arbitrary spatial curvilinear coordinate system have the form [4]

$$t^j{}_{i,j} + \rho f_i = 0$$

$$\left(t^{ij} = -pg^{ij} + 2\frac{\partial\Sigma}{\partial C_{AB}} x^i{}_{,A}\, x^j{}_{,B}; \qquad C_{AB} = g_{ij} x^i{}_{,A} x^j{}_{,B}; \qquad x^i{}_{,A} = \frac{\partial x^i}{\partial X^A} \right) \tag{4.1}$$

Here t^{ij} are the components of the stress tensor, ρ is the density of the deformed medium, f_i are the components of the body forces, p is a Lagrangian multiplier, C_{AB} are the components of the Green tensor, and the g_{ij} are the components of the spatial metric tensor.

For an isotropic material [1 and 4]

$$\Sigma = \Sigma(J_1, J_2) \qquad \left(J_1 = C^A{}_A, \quad J_2 = {}^1/_2\left[J_1{}^2 - C^A{}_B C^B{}_A\right]\right) \tag{4.2}$$

Here J_1 and J_2 are the basic invariants of the Green tensor We introduce the tensor [4]

$$(c^{-1})^{ij} = G^{AB} x^i{}_{,A} x^j{}_{,B}$$

where G_{AB} are the components of the material metric tensor.

Passing from the X^A and x^i coordinates to the convective coordinate system,

$$z^1 = \alpha, \quad z^2 = \beta, \quad z^3 = x^3 = X^3 \tag{4.3}$$

the stress tensor (4.1) is represented in the Green-Zerna form for an isotropic medium [4 and 5] as

$$t^{ij} = -pg^{ij} + 2\frac{\partial\Sigma}{\partial J_1}G^{ij} + 2\frac{\partial\Sigma}{\partial J_2}(J_1G^{ij} - G^{ik}g_{kl}G^{lj}) \qquad (4.4)$$

For the axially symmetric problem in the absence of body forces the equilibrium condition in the convective coordinate system may be written in the form

$$(p\delta_i^j)_{,j} = \left[2\frac{\partial\Sigma}{\partial J_1}G^{jk}g_{ki} + 2\frac{\partial\Sigma}{\partial J_2}(J_1G^{jk}g_{ki} - G^{nk}g_{ni}g_{kl}G^{lj})\right]_{,j} \qquad (i = 1, 2) \quad (4.5)$$

Here

$$G_{11} = (1 + \omega_{\alpha\beta})^2 + \frac{\omega_{\alpha\alpha}{}^2}{2(\beta - \omega_\alpha)}, \qquad G_{12} = (1 + \omega_{\alpha\beta})\,\omega_{\beta\beta} - \frac{\omega_{\alpha\alpha}(1 - \omega_{\alpha\beta})}{2(\beta - \omega_\alpha)}$$

$$G_{22} = \omega_{\beta\beta}{}^2 + \frac{(1 - \omega_{\alpha\beta})^2}{2(\beta - \omega_\alpha)}, \qquad G_{33} = 2(\beta - \omega_\alpha), \quad G_{13} = 0, \quad G_{23} = 0$$

$$g_{11} = (1 - \omega_{\alpha\beta})^2 + \frac{\omega_{\alpha\alpha}{}^2}{2(\beta + \omega_\alpha)}, \qquad g_{12} = -\omega_{\beta\beta}(1 - \omega_{\alpha\beta}) + \frac{\omega_{\alpha\alpha}(1 + \omega_{\alpha\beta})}{2(\beta + \omega_\alpha)}$$

$$g_{22} = \omega_{\beta\beta}^2 + \frac{(1 + \omega_{\alpha\beta})^2}{2(\beta + \omega_\alpha)}, \qquad g_{33} = 2(\beta + \omega), \quad g_{13} = 0, \quad g_{23} = 0$$

Upon calculating the second covariant derivative of the scalar function p, we obtain the condition of compatibility of stress and strain for an isotropic medium with an energy density $\Sigma = \Sigma(J_1, J_2)$

$$\begin{aligned}&[KG^{jk}g_{k1} + L(J_1G^{jk}g_{k1} - G^{nk}g_{n1}g_{kl}G^{lj})]_{,j2} = \\ &= [KG^{jk}g_{k2} + L(J_1G^{jk}g_{k2} - G^{nk}g_{n2}g_{kl}G^{lj})]_{,j1}\end{aligned} \qquad \left(K = \frac{\partial\Sigma}{\partial J_1}, \; L = \frac{\partial\Sigma}{\partial J_2}\right) \qquad (4.6)$$

Only the displacement function $\omega(\alpha,\beta)$ and constants characterizing the mechanical properties of the medium enter into Equation (4.6). Thus, by solving (4.6) we find $\omega(\alpha,\beta)$ and isolate from the combinations of strains those systems which satisfy the equilibrium conditions for an element of the medium.

5. The exposition permits setting up a way of solving axially symmetric problems. Equation (4.6), when integrated, determines a system of displacement functions $\omega(\alpha,\beta)$ compatible with the equilibrium conditions for an element of the medium for a given physical law of relation between stresses and strains.

For this $\omega(\alpha,\beta)$ system, p may be found from (4.5) since the system (4.5) is completely integrable. Thus the components t^{ij} of the stress tensor are determined.

It remains to satisfy the boundary conditions. Evidently the edge conditions, with displacements given on the boundary, may be satisfied in a preliminary way by a choice of $\omega(\alpha,\beta)$ in the corresponding form. Thus there remain the edge conditions in the form

$$t_i^j n_j = t_i \qquad (5.1)$$

where n_i are the components of the unit vector of the external normal to the bounding surface of the deformed medium, and t_i are the components of the external force in respect to unit area of the same surface.

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REMARK CONCERNING THE PAPER "MAXIMUM RANGE FOR A ROCKET IN HORIZONTAL FLIGHT" (PMM Vol.27, № 3, 1963)

(ZAMECHANIE K RABOTE "O MAKSIMAL'NOI DAL'NOSTI POLETA RAKETY V GORIZONTAL'NOI PLOSKOSTI")

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The problem considered in [1] belongs to a class of problems studied in [2]. It is shown there, that the assumption of monotonic behavior of $m(V)$, on which is based the unique optimum control with not more than two switchings, is satisfied for the realistic drag laws

$$D = AV^2 + B\frac{L^n}{V^{2n-2}} \qquad (n = 2 \quad \text{or} \quad n = {}^3/_2)$$

Other aspects of the problem may be found in [3 and 4].

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